Static Stability in Symmetric and Population Games

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Static stability in strategic games differs from dynamic stability in only considering the players' incentives to change their strategies. It does not rely on any assumptions about the players' reactions to these incentives and it is thus independent of the law of motion (e.g., whether players move simultaneously or sequentially). Examples of static notions of stability include evolutionarily stable strategy (ESS) and continuously stable strategy (CSS), both of which are meaningful or justifiable only for particular classes of symmetric and population games, such as games with multilinear payoff functions or with unidimensional strategy spaces. This paper presents a general notion of static stability in symmetric N-player games and population games with non-discrete strategy spaces, of which ESS and CSS are essentially special cases. *JEL Classification:* C72.

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1 Introduction

A strategic game is at equilibrium when the players do not have any incentives to act differently than they do. In other words, at an equilibrium point, no player can increase his payoff by unilaterally changing his strategy. Stability differs in referring to the effects – either on the players' incentives or on the actual strategy choices – of starting at *another*, usually close-by, point. Notions of stability that only examine incentives may be broadly classified as static, and those that look at the consequent changes of strategies may be referred to as dynamic. (For a brief review of some additional notions of stability in strategic games, which fit neither of these categories, see Appendix A.) Dynamic stability relies on a specific law of motion, such as the replicator dynamics. It thus depends both on the game itself, that is, on the payoff functions, and on the choice of dynamics. Static stability, by contrast, depends only on intrinsic properties of the game, and is hence arguably the more basic, fundamental concept. This is not an assessment of the relative importance of the two kinds of stability, but of the logical relation between them.

This paper presents a notion of static stability in symmetric N-player games and in population games that is universal in that it does not depend on structures or properties (such as multilinearity of the payoff function) that only certain kinds of games have. Stability may be local or global. In the former case, it refers to a designated topology on the strategy

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space, which gives a meaning to a *neighborhood* of a strategy. In this paper, there are no restrictions on the choice of topology, which may in particular be the trivial topology, where the only neighborhood of any strategy is the entire strategy space. The latter choice corresponds to global stability, and it is most natural in the case of finite strategy spaces.

The definition of (static) stability in this paper is based on a very simple idea, namely, examination of the incentive to switch from a strategy y to another strategy x in all states in which only these two strategies are used, so that some of the players use x and the others use y. Stability of y means that, for every x in a neighborhood of y, these incentives are negative on average. Formally, this definition represents minimal divergence from the (Nash) equilibrium concept. However, the latter does not imply stability and, in general, it is also not implied by it. The paper's focus is on stable equilibrium strategies, which satisfy both conditions. For a number of large, important classes of symmetric and population games, it examines the specific meaning of stability (in the above sense) in the class. In some cases, the latter turns out to be equivalent, or essentially so, to an established "native" notion of stability. Evolutionary stability for symmetric $n \times n$ games and continuous stability for games with unidimensional strategy spaces are examples of this. The definition outlined above thus turns static stability from a generic notion to a concrete, well-defined one.

The reliance of static stability solely on incentives makes it particularly suitable for use in comparative statics analysis. In particular, in turns out that the welfare effects of altruism and spite, that is, whether people in a group where everyone shares such sentiments are likely to fare better or worse than in a group where people are indifferent to the others' payoffs, depend on the static stability or instability of the corresponding equilibrium strategies (Milchtaich, 2012). Such a connection between stability and comparative statics is akin to Samuelson's (1983) "correspondence principle".

2 Symmetric and population games

A *symmetric N-player game* ($N \geq 1$) is a real-valued (payoff[1]) function $g: X^N \to \mathbb{R}$ that is defined on the N-times Cartesian product of a (finite or infinite nonempety) set X, the players' common *strategy space*, and is invariant to permutations of its second through Nth arguments. If one player uses strategy x and the others use $y, z, \dots, w$, in any order, the first player's payoff is $g(x, y, z, \dots, w)$. A strategy y is a (symmetric Nash) *equilibrium strategy* in g, with the *equilibrium payoff* $g(y, y, \dots, y)$, if it is a best response to itself: for every strategy x,

$$g(y, y, \dots, y) \geq g(x, y, \dots, y). \tag{1}$$

A *population game*, as defined in this paper, is formally a symmetric two-player game g such that the strategy space X is a convex set in a (Hausdorff real) linear topological space (for example, the unit simplex in a Euclidean space) and $g(x, y)$ is continuous in y for all $x \in X$. However, the game is interpreted not as representing an interaction between two

[1] In this paper, the payoff function and the game itself are identified.

specific players but as one involving an (effectively) infinite[2] population of individuals who are "playing the field". This means that an individual's payoff $g(x, y)$ depends only on his own strategy x and on the *population strategy* y. The latter may be, for example, the population's *mean* strategy with respect to some nonatomic measure, which attaches zero weight to each individual. In this case, the meaning of the equilibrium condition

$$g(y, y) = \max_{x \in X} g(x, y) \tag{2}$$

is that, in a *monomorphic* population, where everyone plays y, single individuals cannot increase their payoff by choosing any other strategy. Alternatively, a population game g may describe a dependence of an individual's payoff on the *distribution* of strategies in the population (Bomze and Pötscher, 1989), with the latter expressed by the population strategy y. In this case, X consists of *mixed strategies*, that is, probability measures on some underlying space of allowable actions or (pure[3]) strategies, and $g(x, y)$ is linear in x and expresses the expected payoff for an individual whose choice of strategy is random with distribution x. Provided the space X is rich enough, the equilibrium condition (2) now means that the population strategy y is supported in the collection of all best response pure strategies. In other words, the (possibly) *polymorphic* population is in an equilibrium state.

Example 1. *Random matching in a symmetric multilinear game* (Bomze and Weibull, 1995; Broom et al., 1997). The N players in a symmetric N-player game g are picked up independently and randomly from an infinite population of potential players. The strategy space X is a convex set in a linear topological space, and g is continuous and is linear in each of its arguments. (This assumption may be relaxed by not requiring linearity in the first argument.) Because of the multilinearity of g, a player's expected payoff only depends on his own strategy x and on the population's mean strategy y. Specifically, it is given by

$$\bar{g}(x, y) \stackrel{\text{def}}{=} g(x, y, \dots, y). \tag{3}$$

This defines a population game $\bar{g}: X^2 \to \mathbb{R}$. Clearly, a strategy y is an equilibrium strategy in $\bar{g}$ if and only if it is an equilibrium strategy in the underlying N-player game g.

Example 2. *Nonatomic congestion game.* An infinite population of identical users, modeled as the unit interval $[0,1]$, shares a finite number n of common resources (for example, road segments). The cost of using each resource j (for example, the time is takes to traverse the road) depends on the size of the set of its users. This dependence is specified by a continuous and strictly increasing cost function $c_j: [0, \infty) \to [0, \infty)$. Each user t has to choose a subset of resources (for example, a route, comprising several road segments), which can be expressed as a binary vector $\sigma(t) = (\sigma_1(t), \sigma_2(t), \dots, \sigma_n(t))$, where $\sigma_j(t) = 1$ or 0 indicates that resource j is included or is not included in t's choice, respectively. The

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[2] An infinite population may represent the limiting case of an increasingly large population, with the possible effect of each player on each of the others correspondingly decreasing. Alternatively, it may represent all possible characteristics of players, or *potential* players, when the number of *actual* players is finite.

[3] 'Pure' and 'mixed' are relative terms. In particular, a pure strategy may itself be a probability vector.

vector must belong to a specified finite collection $\check{X} \subseteq \{0,1\}^n$, which describes the allowable subsets of resources (for example, all routes from town A to town B). The population strategy y is the users' mean choice:

$$y = \int_0^1 \sigma(t)\, dt.$$

It lies in the convex hull of $\check{X}$,

$$X \stackrel{\text{def}}{=} \operatorname{co} \check{X} \subseteq \mathbb{R}^n,$$

and it is well-defined if for each j the set $\{0 \le t \le 1 \mid \sigma_j(t) = 1\}$ is measurable. The population strategy determines the cost of each allowable subset of resources, and more generally, the cost of each (mixed) strategy $x = (x_1, x_2, \dots, x_n) \in X$. Specifically, the negative of the latter, which is the payoff $g(x, y)$, is given by

$$g(x,y) = -\sum_{j=1}^{n} x_j c_j(y_j). \tag{4}$$

This defines a population game $g: X^2 \to \mathbb{R}$.

3 Static stability

By far the most well-known kind of static stability in symmetric two-player games and population games is evolutionary stability (Maynard Smith, 1982).

Definition 1. A strategy y in a symmetric two-player game or population game g is an *evolutionarily stable strategy* (ESS) or a *neutrally stable strategy* (NSS) if, for every strategy $x \neq y$, for sufficiently small[4] $\epsilon > 0$ the inequality

$$g(y, \epsilon x + (1-\epsilon)y) > g(x, \epsilon x + (1-\epsilon)y) \tag{5}$$

or a similar weak inequality, respectively, holds. An ESS or NSS *with uniform invasion barrier* satisfies the stronger condition obtained by interchanging the two logical quantifiers: For sufficiently small $\epsilon > 0$ (which cannot vary with x), (5) or a similar weak inequality, respectively, holds for all $x \neq y$.

Continuous stability (Eshel and Motro, 1981; Eshel, 1983) is another kind of static stability, which is applicable to games with a unidimensional strategy space.

Definition 2. In a symmetric two-player game or population game g with a strategy space that is a (finite or infinite) interval in the real line $\mathbb{R}$, an equilibrium strategy y is a *continuously stable strategy* (CSS) if it has a neighborhood where, for every strategy $x \neq y$, for sufficiently small $\epsilon > 0$ the inequality

$$g(x + \epsilon(y - x), x) > g(x, x) \tag{6}$$

holds and a similar inequality with ϵ replaced by $-\epsilon$ does not hold.

[4] A condition holds for "sufficiently small" $\epsilon > 0$ if there is some $\delta > 0$ such that the condition holds for all $0 < \epsilon < \delta$.

In other words, a strategy y that satisfies the “global” condition of being an equilibrium strategy[5] is a CSS if it also satisfies the “local” condition (known as m-stability or convergence stability; Taylor, 1989; Christiansen, 1991) that every nearby strategy x is not a best response to itself, specifically, any small deviation from x towards y, but not in the other direction, increases the payoff.

Yet another static notion of stability in symmetric and (with $N = 2$) population games is local superiority (or strong uninvadability; Bomze, 1991).

Definition 3. A strategy y in a symmetric N-player game or population game g is *locally superior* if it has a neighborhood where, for every strategy $x \neq y$,

$$g(y, x, \dots, x) > g(x, x, \dots, x). \tag{7}$$

Local superiority is applicable to any symmetric or population game in which the strategy space is a topological space, so that the notion of neighborhood is well defined.[6] It does not rely on any other properties of the strategy space or of the payoff function – unlike ESS and CSS, which would not be meaningful without a linear structure. It is well known (see Section 6) that in the special case of symmetric $n \times n$ games local superiority is in fact equivalent to the ESS condition. However, for games with a unidimensional strategy space (Section 7) local superiority and CSS are not equivalent.

The next section presents a universal notion of static stability, that is, one which is applicable to all symmetric and population games. It (essentially) gives ESS and CSS as special cases when applied to specific classes of games.

4 A general framework

Inequality (1) in the equilibrium condition and inequality (7) in the definition of local superiority both concern a player’s lack of incentive to use a particular alternative x to his strategy y. In the equilibrium condition, all the other players are using y, and in local superiority, they all use x. Stability, as defined below, differs from both concepts in considering not only the incentives to be first or last to move from y to x but also all the intermediate cases. In the simplest version, which is given by the followings definition (and is extended in Section 4.2), the same weight is attached to all cases. Put differently, stability requires that, when the players move one-by-one to x from y, the corresponding changes of payoff are negative *on average*.

Definition 4. A strategy y in a symmetric N-player game $g: X^N \to \mathbb{R}$ is *stable*, *weakly stable* or *definitely unstable* if it has a neighborhood where, for every strategy $x \neq y$, the inequality

$$\frac{1}{N}\sum_{j=1}^{N}\big(g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(y, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}})\big) < 0, \tag{8}$$

a similar weak inequality or the reverse (strict) inequality, respectively, holds.

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[5] The original definition of CSS differs slightly from the version given here in requiring a stronger global condition, which is a version of ESS.

[6] A subset of X is a *neighborhood* of a strategy x if its interior includes x (Kelley, 1955).

Stability, as defined here, is a local concept. It refers to neighborhood systems of strategies or equivalently to a topology on the strategy space X. The topology may be explicitly specified or it may be understood from the context. The latter applies when X may be naturally viewed as a subspace of a Euclidean space or some other standard topological space, so that its topology is the relative one. For example, if the strategy space is an interval in the real line $\mathbb{R}$, so that strategies are simply (real) numbers, a set of strategies is a neighborhood of a strategy y if and only if, for some $\varepsilon > 0$, every $x \in X$ with $|x - y| < \varepsilon$ is in the set. In a game with a finite number of strategies, it may seem natural to consider the discrete topology, that is, to view strategies as isolated. However, as discussed in Section 5 below, a more useful choice of topology in a finite game is the trivial, or indiscrete, topology. This choice effectively puts topology out of the way, since the only neighborhood of any strategy is the entire strategy space. The trivial topology may be used also with an infinite X. Stability, weak stability or definite instability of a stategy y with respect to the trivial topology automatically implies the same with respect to any other topology. Such a strategy y will be referred to as *globally* stable, weakly stable or definitely unstable, respectively.

In some classes of games (see Section 6, 8 and 9), stability of a strategy automatically implies that it is an equilibrium strategy. However, in general, neither of these conditions implies the other. The difference is partially due to equilibrium being a global condition: all alternative strategies, not only nearby ones, are considered. However, is persists if 'equilibrium' is replaced by 'local equilibrium' (with the obvious meaning), and also if the strategy space has the trivial topology, which obviates the distinction between local and global. A *stable equilibrium strategy* is a strategy that satisfies both conditions. It is not difficult to see that in the special case of a symmetric two-player game, where the equilibrium condition can be written as (2) and (8) can be rearranged to read

$$\frac{1}{2}\big(g(x,x) - g(y,x) + g(x,y) - g(y,y)\big) < 0, \tag{9}$$

a strategy y is a stable equilibrium strategy if and only if it has a neighborhood where, for every $x \neq y$, the inequality

$$pg(x,x) + \ (1-p)g(x,y) < pg(y,x) + (1-p)g(y,y) \tag{10}$$

holds for all $0 < p \leq 1/2$. This condition means that the alternative strategy x affords a lower expected payoff than y against an uncertain strategy that may be x or y, with the former no more likely than the latter.

Local superiority is similar to stability in being a local condition. And for equilibrium strategies in symmetric two-player games locally superiority implies stability, since (with $N = 2$) inequality (9) can be obtained by averaging (1) and (7). The same is true for certain kinds of symmetric games with more than two players (see Section 9). However, even for equilibrium strategies in two-player games, the reverse implication does not generally hold (see Section 7).

4.1 Stability in population games

Stability in population games can be defined by a variant of Definition 4 that replaces the number of players using strategy x or y with the size of the subpopulation to which the strategy applies, p or $1-p$ respectively. Correspondingly, the sum in (8) is replaced with an integral.

Definition 5. A strategy y in a population game $g: X^2 \to \mathbb{R}$ is *stable*, *weakly stable* or *definitely unstable* if it has a neighborhood where, for every strategy $x \neq y$, the inequality

$$\int_0^1 \big(g(x, px + (1-p)y) - g(y, px + (1-p)y)\big)\, dp < 0, \tag{11}$$

a similar weak inequality or the reverse (strict) inequality, respectively, holds.

The difference between stability in the sense of Definition 5 and in the sense of (the two versions of) ESS (Definition 1) boils down to a different meaning of proximity between population strategies. The definition of ESS reflects the view that a population strategy is close to y if the latter applies to a large subpopulation, of size $1-\epsilon$, and another strategy x applies to a small subpopulation, of size ϵ. By contrast, in Definition 5, the subpopulation to which x applies need not be small, but x itself is assumed close to y. The significance of this difference between the definitions is examined in Sections 6 and 9.

If a population game $\bar{g}$ is derived from a symmetric multilinear game g as in Example 1, then, depending on whether y is viewed as a strategy in g or $\bar{g}$, Definition 4 or 5 applies. However, the point of view turns out to be immaterial.

Proposition 1. A strategy y in a symmetric multilinear N-player game g is stable, weakly stable or definitely unstable if and only if it has the same property in the population game $\bar{g}$ defined by (3).

Proof. For $0 \le p \le 1$ and strategies x, y and

$$x_p = px + (1-p)y,$$

the linearity of g in each of its second through Nth arguments and its invariance to permutations of these arguments give

$$\begin{aligned} \bar{g}(x, x_p) - \bar{g}(y, x_p) &= g(x, x_p, \dots, x_p) - g(y, x_p, \dots, x_p) \\ &= \sum_{j=1}^{N} B_{j-1,N-1}(p)\big(g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(y, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}})\big), \end{aligned} \tag{12}$$

where

$$B_{j-1,N-1}(p) = \binom{N-1}{j-1} p^{j-1}(1-p)^{N-j}, \qquad j = 1,2, \dots, N$$

are the Bernstein polynomials. These polynomials satisfy the equalities

$$\int_0^1 B_{j-1,N-1}(p)\, dp = \frac{1}{N}, \qquad j = 1,2, \dots, N. \tag{13}$$

It therefore follows from (12) by integration that the expression obtained by replacing g on the left-hand side of (11) by $\bar{g}$ is equal to the expression on the left-hand side of (8). ■

In the subsequent sections, Definitions 4 and 5 are applied, or restricted, to a number of specific classes of symmetric and population games and the results are compared with certain “native” notions of stability for these games. The rest of the present section is concerned with an extension of the above framework, which facilitates the capturing of some additional native notions of stability.

4.2 $\bar{p}$-stability

Stability as defined above in a sense occupies the midpoint between equilibrium and local superiority. It takes into consideration a player’s incentive to be the first or last to switch to a particular alternative strategy, but attaches to these extreme cases the same weight it attaches to each of the intermediate ones. This uniform distribution of weights may be interpreted as expressing a particular belief of the player about the total number of players who will be using the alternative strategy after he switches to it, with the rest using the original strategy. Namely, the probabilities $p_1, p_2, \dots, p_N$ that that number is $1,2, \dots, N$ are all equal:

$$p_j = \frac{1}{N}, \qquad j = 1,2, \dots, N. \tag{14}$$

Thus, unlike local superiority, in which the gain from switching from strategy y to the alternative strategy x is computed under the belief that all the other players are using x, in stability the expected gain is with respect to the probabilities (14), which give the expression on the left-hand side of (8). A straightforward generalization of both concepts is to allow *any* beliefs.

Definition 6. For a probability vector $\bar{p} = (p_1, p_2, \dots, p_N)$, a strategy y in a symmetric N-player game $g: X^N \to \mathbb{R}$ is *$\bar{p}$-stable*, *weakly $\bar{p}$-stable* or *definitely $\bar{p}$-unstable* if it has a neighborhood where, for every strategy $x \neq y$, the inequality

$$\sum_{j=1}^{N} p_j (g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(y, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}})) < 0, \tag{15}$$

a similar weak inequality or the reverse (strict) inequality, respectively, holds.

If each of the other players switches to x with probability p and stays with y with probability $1 - p$, then, depending on whether the choices are perfectly correlated (i.e., identical) or independent, respectively,

$$p_j = \begin{cases} 1 - p, & j = 1 \\ 0 & , j = 2,3, \dots, N-1 \\ p & , j = N \end{cases} \tag{16}$$

or

$$p_j = \binom{N-1}{j-1} p^{j-1} (1-p)^{N-j} = B_{j-1,N-1}(p), \qquad j = 1,2, \dots, N. \tag{17}$$

A strategy y is *dependently-* or *independently-stable* if it is $\bar{p}$-stable with $\bar{p} = (p_1, p_2, \dots, p_N)$ given by (16) or (17), respectively, for all $0 < p < 1$.

The number of other players using strategy x and the number using y have a symmetric joint distribution if the two numbers are equally distributed (and, in particular, have an equal expectation of $(N-1)/2$), that is,

$$p_j = p_{N-j+1}, \qquad j = 1,2, \dots, N. \tag{18}$$

For $\bar{p} = (p_1, p_2, \dots, p_N)$ that satisfies (18), the left-hand side of (15) is equal to the more symmetrically-looking expression

$$G_{\bar{p}}(x,y) \stackrel{\text{def}}{=} \sum_{j=1}^{n} p_j \big(g(\underbrace{x, \dots, x}_{j \text{ times}}, y, \dots, y) - g(\underbrace{y, \dots, y}_{j \text{ times}}, x, \dots, x)\big).$$

Thus, a strategy y is $\bar{p}$-stable if and only if it has a neighborhood where it is the unique best response to itself in the symmetric two-player zero-sum game $G_{\bar{p}}: X^2 \to \mathbb{R}$. As a special case, this characterization applies to stability, that is, to $\bar{p}$ given by (14). A strategy y is *symmetrically-stable* if it is $\bar{p}$-stable for all $\bar{p}$ satisfying (18).

For single-player games ($N = 1$), stability and $\bar{p}$-stability of a strategy mean the same thing, namely, strict local optimality: switching to any nearby alternative strategy reduces the payoff. For $N = 2$, stability does not generally imply $\bar{p}$-stability (or vice versa) but the implication does partially hold (specifically, holds whenever $0 < p_2 \le 1/2$) in the case of an equilibrium strategy (see (10)). To fully appreciate the differences between stability in the sense of Definition 4 and the varieties based on $\bar{p}$-stability it is necessary to look at multiplayer games. One class of such games is examined in Section 9.

5 Finite games and risk dominance

In every symmetric or population game, every isolated strategy is trivially stable. Therefore, if the strategy space X has the discrete topology, that is, all singletons are open sets, then all strategies are stable. The definition of stability is therefore of interest only for games with non-discrete strategy spaces. This includes games with a finite number of strategies where the topology on X is the trivial one, so that stability and definite instability mean *global* stability and definite instability (see Section 4). The simplest (interesting) such game is a finite symmetric two-player game with only two strategies, strategy 1 and strategy 2, for example, the game with payoff matrix

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 3 & 4 \\ 1 & 5 \end{pmatrix} \end{array}$$

(where the rows correspond to the player's own strategy and the columns to the opponent's strategy). In this example, both strategies are equilibrium strategies. Strategy 1 is stable (in the sense of Definition 4, in the form (9)) and strategy 2 is definitely unstable, since

$$\frac{1}{2}(5 - 4 + 1 - 3) < 0 < \frac{1}{2}(3 - 1 + 4 - 5).$$

The two inequalities, which are clearly equivalent, have an additional meaning. Namely, they express the fact that $(1,1)$ is the *risk dominant* equilibrium (Harsanyi and Selten, 1988). It is not hard to see that this coincidence of stability and risk dominance holds in general – it is not a special property of the payoffs in this example.

Proposition 2. In a finite symmetric two-player game with two strategies, an equilibrium strategy y is globally stable if and only if the equilibrium (y, y) is risk dominant.

For a pure equilibrium strategy y, risk dominance of (y, y) is equivalent to global stability of y also in the mixed extension g of the finite game, that is, when mixed strategies are allowed. This follows from the fact that global stability of y in the finite game implies that inequality (10) holds for all $0 < p \leq 1/2$, where x is the other pure strategy. Because of the bilinearity of g, the same is then true with x replaced by any convex combination of x and y other than y itself, which proves that y is globally stable in g. However, since in the mixed extension of the finite game, which is a symmetric 2×2 game, the strategy space X is essentially the unit interval, the natural topology on X is not the trivial topology but the usual one. Stability with respect to the latter is a weaker condition than global stability. For example, as shown in the next section, it holds for *both* pure strategies if (as in the above example) the corresponding strategy profiles are strict equilibria.

6 Symmetric $n \times n$ games and evolutionary stability

A symmetric $n \times n$ game is given by a (square) payoff matrix A with these dimensions. The strategy space X, whose elements are referred to as mixed strategies, is the unit simplex in $\mathbb{R}^n$. The interpretation is that there are n possible actions, and a strategy $x = (x_1, x_2, \dots, x_n)$ is a probability vector specifying the probability x_i with which each action i is used ($i = 1,2, \dots, n$). The set of all actions i with $x_i > 0$ is the *support* of x. A strategy is *pure* or *completely mixed* if its support contains only a single action i (in which case the strategy itself may also be denoted by i) or all n actions, respectively. The game (i.e., the payoff function) $g: X^2 \to \mathbb{R}$ is defined by

$$g(x, y) = x^{\mathrm{T}} A y$$

(where superscript T denotes transpose and strategies are viewed as column vectors). Thus, g is bilinear, and $A = \big(g(i,j)\big)_{i,j=1}^{n}$.

A symmetric $n \times n$ game may be viewed either as a symmetric two-player game or as a population game. In the former case, Definition 4 applies, and in the latter, Definition 5. However, by Proposition 1, the two definitions of stability in fact coincide, and the same is true for weak stability and for definite instability. Moreover, as the next two results show, stability is also equivalent to evolutionary stability and to local superiority (see Section 3). It also follows from these results that every (even weakly) stable strategy in a symmetric $n \times n$ game is an equilibrium strategy, and every strict equilibrium strategy is stable.

The following proposition is rather well known (Bomze and Pötscher, 1989; van Damme, 1991, Theorem 9.2.8; Weibull, 1995, Propositions 2.6 and 2.7; Bomze and Weibull, 1995).

Proposition 3. For a strategy y in a symmetric $n \times n$ game g, the following conditions are equivalent:[7]

(i) Strategy y is an ESS.
(ii) Strategy y is an ESS with uniform invasion barrier.
(iii) For every strategy $x \neq y$ in some neighborhood of y,

$$g(y,x) > g(x,x). \tag{19}$$

(iv) For every $x \neq y$, the (weak) inequality $g(y,y) \geq g(x,y)$ holds, and if it holds as equality, then (19) also holds.

An NSS is characterized by similar equivalent conditions, in which the strict inequality (19) is replaced by a weak one.

A completely mixed equilibrium strategy y in a symmetric $n \times n$ game g is said to be *definitely evolutionarily unstable* (Weissing, 1991) if the reverse inequality to that in (19) holds for all $x \neq y$.

Theorem 1. A strategy y in a symmetric $n \times n$ game g is stable or weakly stable if and only if it is an ESS or an NSS, respectively. A completely mixed equilibrium strategy is definitely unstable if and only if it is definitely evolutionarily unstable.

Proof. The two inequalities in (iii) and (iv) of Proposition 3 together imply (9), and the same is true with the strict inequalities (9) and (19) both replaced by their weak versions. This proves that a sufficient condition for stability or weak stability of a strategy y is that it is an ESS or an NSS, respectively. For a completely mixed equilibrium strategy y, the inequality in (iv) automatically holds as equality for all x, and therefore a similar argument proves that a sufficient condition for definite instability of y is that it is definitely evolutionarily unstable.

In remains to prove necessity. For a stable strategy y, (9) holds for all nearby strategies $x \neq y$. Therefore, y has the property that, for *every* strategy $x \neq y$, for sufficiently small $\varepsilon > 0$

$$\begin{aligned} g(\varepsilon x + (1-\varepsilon)y, \varepsilon x + (1-\varepsilon)y) - g(y, \varepsilon x + (1-\varepsilon)y) \\ + g(\varepsilon x + (1-\varepsilon)y, y) - g(y,y) < 0. \end{aligned} \tag{20}$$

It follows from the bilinearity of g that (20) is equivalent to

$$(2-\varepsilon)(g(y,y) - g(x,y)) + \varepsilon(g(y,x) - g(x,x)) > 0. \tag{21}$$

Therefore, the above property of y is equivalent to (iv) of Proposition 3, which proves that y is an ESS. Similar arguments show that a weakly stable strategy is an NSS and that a definitely unstable completely mixed equilibrium strategy is definitely evolutionarily unstable. In the first case, the proof needs to be modified only by replacing the strict inequalities in (19), (20) and (21) by weak inequalities, and in the second case (in which the first term in (21) vanishes for all x), they need to be replaced by the reverse inequalities. ■

[7] Note that condition (iii) means that y is locally superior and that the first part of (iv) means that it is an equilibrium strategy.

7 Games with a unidimensional strategy space and continuous stability

In a symmetric two-player game or population game where the strategy space is an interval in the real line $\mathbb{R}$, stability or instability of an equilibrium strategy, in the sense of either Definition 4 or 5, has a simple, familiar meaning. As shown below, if the payoff function is twice continuously differentiable and with the possible exception of certain borderline cases, the equilibrium strategy is stable or definitely unstable if, at the (symmetric) equilibrium point, the graph of the best-response function, or reaction curve, intersects the forty-five degree line from above or below, respectively. This geometric characterization of stability and its differential counterpart are also shared by continuous stability (Section 3), which shows that these two notions of static stability are essentially equivalent.

Theorem 2. Let g be a symmetric two-player game or population game with a strategy space X that is a (finite or infinite) interval in the real line $\mathbb{R}$, and y an interior equilibrium strategy (that is, one lying in the interior of X) such that g has continuous second-order partial derivatives[8] in a neighborhood of the equilibrium point (y, y). If

$$g_{11}(y,y) + g_{12}(y,y) < 0, \tag{22}$$

then y is stable and a CSS. If the reverse inequality holds, then y is definitely unstable and not a CSS.

Proof. Using Taylor's theorem, it is not difficult to show that, for x tending to y, the left-hand sides of both (9) and (11) can be expressed as

$$g_1(y,y)(x-y) + \frac{1}{2}\big(g_{11}(y,y) + g_{12}(y,y)\big)(x-y)^2 + o((x-y)^2). \tag{23}$$

Since y is an interior equilibrium strategy, the first term in (23) is zero. Therefore, a sufficient condition for (23) to be negative or positive for all $x \neq y$ in some neighborhood of y (and hence for y to be stable or definitely unstable, respectively) is that $g_{11}(y,y) + g_{12}(y,y)$ has that sign.

Next, consider the CSS condition in Definition 2. It may be possible to determine whether this condition holds by looking at the sign of

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} \big(g(x,x) - g(x+\epsilon(y-x),x)\big) = g_1(x,x)(x-y). \tag{24}$$

For x tending to y, (24) is given by an expression similar to (23) except that it lacks the factor $1/2$. Therefore, if (22) or the reverse inequality holds, then (24) is negative or positive, respectively, for all $x \neq y$ in some neighborhood of y. In the first or second case, (6) holds or does not hold, respectively, for $\epsilon > 0$ sufficiently close to 0 and the converse is true for $\epsilon < 0$. Therefore, in the first case, y is a CSS, and in the second case, it is not a CSS. ∎

[8] Partial derivatives are denoted by subscripts. For example, g_{12} is the mixed second-order partial derivative of g.

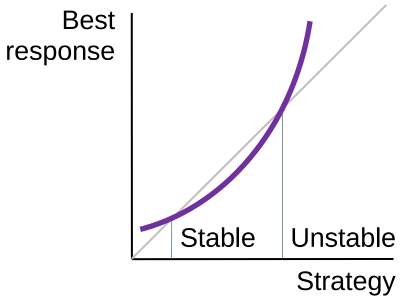

Figure 1. An equilibrium strategy is stable (and a CSS) or definitely unstable (and not a CSS) if, at the equilibrium point, the reaction curve (thick line) intersects the forty-five degree line (thin) from above or below, respectively.

The connection between inequality (22) and the slope of the reaction curve can be established as follows (Eshel, 1983). If y is an interior equilibrium strategy, then it follows from the equilibrium condition (2) that $g_1(y, y) = 0$ and $g_{11}(y, y) \leq 0$. If the last inequality is in fact strict, then by the implicit function theorem there is a continuously differentiable function f from some neighborhood of y to the strategy space, with $f(y) = y$, such that $g_1(f(x), x) = 0$ and $g_{11}(f(x), x) < 0$ for all strategies x in the neighborhood. Thus, strategy $f(x)$ is a local best response to x. By the chain rule, at the point y

$$f'(y) = -\frac{g_{12}(y, y)}{g_{11}(y, y)}.$$

Therefore, (22) holds (so that y is stable) or the reverse inequality holds (y definitely unstable) if and only if the slope of the function f at y is less or greater than 1, respectively.[9] In the first case, the reaction curve (see Figure 1), which is the graph of f, intersects the forty-five degree line from above (which means that the (local) fixed point index is $+1$; see Dold, 1980). In the second case, the intersection is from below (and the fixed point index is -1).

In a population or symmetric two-player game with a unidimensional strategy space, an equilibrium strategy y that is locally superior is said to be a *neighborhood invader strategy* (Apaloo, 1997). As shown (see Section 4), such a strategy is necessarily stable. However, unlike for symmetric $n \times n$ games (Section 6), the converse is false. This can be seen most easily by considering the differential condition for local superiority of an equilibrium strategy y (Oechssler and Riedel, 2002). This condition differs from (22) in that the second term $g_{12}(y, y)$ is multiplied by 2. Since, by the equilibrium condition, the first term $g_{11}(y, y)$ is necessarily nonpositive, this makes the condition more demanding than (22).

8 Potential games

A symmetric N-player game $g: X^N \to \mathbb{R}$ is called an (exact) potential game if it has an (exact) *potential*, which is a symmetric function (that is, a function that is invariant under permutations of its N arguments) $F: X^N \to \mathbb{R}$ such that, whenever a single player changes his strategy, the change in the player's payoff is equal to the change in F. Thus, for any

[9] This geometric condition for static stability is weaker than the corresponding one for dynamic stability, which requires the *absolute value* of slope to be less than 1 (Fudenberg and Tirole, 1995).

$N + 1$ (not necessarily distinct) strategies $x, x', y, z, \dots, w$,

$$F(x, y, z, \dots, w) - F(x', y, z, \dots, w) = g(x, y, z, \dots, w) - g(x', y, z, \dots, w). \tag{25}$$

The potential is unique up to an additive constant. Clearly, a necessarily condition for the existence of a potential is that the total change of payoff of two players who change their strategies one after the other does not depend on the order of their moves: For any $N + 2$ strategies $x, x', y, y', z, \dots, w$,

$$\begin{aligned} &g(x, y, z, \dots, w) - g(x', y, z, \dots, w) + g(y, x', z, \dots, w) - g(y', x', z, \dots, w) \\ &\qquad = g(y, x, z, \dots, w) - g(y', x, z, \dots, w) + g(x, y', z, \dots, w) - g(x', y', z, \dots, w). \end{aligned}$$

It is not difficult to show that this condition is also sufficient (see Monderer and Shapley, 1996, Theorem 2.8, which however refers to general, not necessarily symmetric, games, for which the potential function is also not symmetric). Moreover, if g is the mixed extension of a finite game, then g is a potential game if and only if the above condition holds for (any $N + 2$) pure strategies (Monderer and Shapley, 1996, Lemma 2.10). In this case, the potential, like the game g itself, is multilinear.

Example 3. *Symmetric* 2×2 *games.* Every symmetric 2×2 game g, with pure strategies 1 and 2, is a potential game, since it is easy to see that it satisfies the above condition for pure strategies. It is moreover not difficult to check that the following bilinear function is a potential for g:

$$F(x, y) = (g(1,1) - g(2,1))x_1 y_1 + (g(2,2) - g(1,2))x_2 y_2. \tag{26}$$

The potential F of a symmetric potential game g may itself be viewed as a symmetric N-player game, indeed, a doubly symmetric one.[10] It follows immediately from (25) that F and g have exactly the same equilibrium strategies, stable and weakly stable strategies, and definitely unstable strategies. Stability and instability in this case have a strikingly simple characterization, which follows immediately from the observation that the sum in (8) is equal the difference $F(x, x, \dots, x) - F(y, y, \dots, y)$ divided by N.

Theorem 3. In a symmetric N-player game with a potential F, a strategy y is stable, weakly stable or definitely unstable if and only if it is a strict local maximum point, a local maximum point or a strict local minimum point, respectively, of the function $x \mapsto F(x, x, \dots x)$. If $(y, y, \dots, y)$ is a global maximum point of F itself, then y is in addition an equilibrium strategy.

The following simple result illustrates the theorem. It also makes use of Theorem 1 and Example 3.

Corollary 1. In a symmetric 2×2 game g with pure strategies 1 and 2, a (mixed) strategy is an ESS or an NSS if and only if it is a strict local maximum point or a local maximum point, respectively, of the quadratic function $\Phi: X \to \mathbb{R}$ defined by

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[10] A symmetric game is *doubly symmetric* if it has a symmetric payoff function, which means that the players' payoffs are always equal.

$$\Phi(x) = \frac{1}{2}(g(1,1) - g(2,1))x_1^2 + \frac{1}{2}(g(2,2) - g(1,2))x_2^2. \tag{27}$$

8.1 Potential in population games

For population games, which represent interactions involving many players whose individual actions have negligible effects on the other players, the definition of potential may be naturally adapted by replacing the difference on the left-hand side of (25) with a derivative.

Definition 7. For a population game $g: X^2 \to \mathbb{R}$, a continuous function $\Phi: X \to \mathbb{R}$ is a *potential* if for all $x, y \in X$ and $0 < p < 1$ the following derivative exists and satisfies the equality:

$$\frac{d}{dp}\Phi(px + (1-p)y) = g(x, px + (1-p)y) - g(y, px + (1-p)y). \tag{28}$$

Example 4. *Symmetric* 2×2 *games, viewed as population games.* It is easy to check that, for every such game g, with pure strategies 1 and 2, the function Φ defined by (27) is a potential. Note that, unlike the function F defined in (26), Φ is a function of one variable only.

Example 4 and Corollary 1 hint at the following general result (which, in view of the example, provides an alternative proof for the corollary). As for symmetric games, stability and instability (here, in the sense of Definition 5) of a strategy y in a population game with a potential Φ is related to y being a local extremum point of the potential.

Theorem 4. In a population game g with a potential Φ, a strategy y is stable, weakly stable or definitely unstable if and only if it is a strict local maximum point of Φ, a local maximum point of Φ or a strict local minimum point of Φ, respectively. In the first two cases, y is in addition an equilibrium strategy. If the potential Φ is strictly concave, an equilibrium strategy is a strict global maximum point of Φ, and, necessarily, the game's unique stable strategy.

Proof. By (28), the left-hand side of (11) can be written as

$$\int_0^1 \frac{d}{dp}\Phi(px + (1-p)y)\, dp.$$

This integral equals $\Phi(x) - \Phi(y)$, which proves the first part of the theorem. It also follows from (28), in the limit $p \to 0$, that for all x and y

$$\left.\frac{d}{dp}\right|_{p=0^+} \Phi(px + (1-p)y) = g(x, y) - g(y, y). \tag{29}$$

If y is a local maximum point of Φ, then the left-hand side is nonpositive, which proves that y is an equilibrium strategy.

To prove the last part of the theorem, consider an equilibrium strategy y. For any strategy $x \neq y$, the right-, and hence also the left-, hand side of (29) is nonpositive. If Φ is strictly concave, this implies that the left-hand side of (28) is negative for all $0 < p < 1$, which proves that y is a strict global maximum point of Φ. ∎

Since a potential is by definition a continuous function, an immediate corollary of Theorem 4 is the following result, which concerns the *existence* of (at least) weakly stable strategies. It sheds light on the difference in this respect between symmetric 2×2 games and, for example, 3×3 games. The former, which as indicated are potential games, always have at least one NSS, whereas for the latter, it is well known that this is not so.

Corollary 2. If a population game g with a potential Φ has a compact strategy space, then it has at least one weakly stable strategy. If in addition the number of such strategies is finite, they are all stable.

The term potential is borrowed from physics, where it refers to a scalar field whose gradient gives the force field. Force is analogous to incentive here. The analogy can be taken one step further by presenting the payoff function g as the differential of the potential Φ. This requires Φ to be defined not only on the strategy space X (which by definition is a convex set in a linear topological space) but on its *cone* $\hat{X}$, which is the collection of all space elements that can be written as a strategy x multiplied by a positive number t. For example, if strategies are probability measures, Φ needs to be defined for all positive, non-zero finite measures. The *differential* of the potential can then be defined as its directional derivative, that is, as the function $d\Phi: \hat{X}^2 \to \mathbb{R}$ given by[11]

$$d\Phi(\hat{x}, \hat{y}) = \frac{d}{dt}\bigg|_{t=0^+} \Phi(t\hat{x} + \hat{y}). \tag{30}$$

The differential exists if the (right) derivative in (30) exists for all $\hat{x}, \hat{y} \in \hat{X}$.

Proposition 4. Let $g: X^2 \to \mathbb{R}$ be a population game and $\Phi: \hat{X} \to \mathbb{R}$ a continuous function (on the cone of the strategy space) such that $d\Phi: \hat{X}^2 \to \mathbb{R}$ exists, is continuous in the second argument and satisfies

$$d\Phi(x, y) = g(x, y), \qquad x, y \in X. \tag{31}$$

Then the restriction of Φ to X is a potential for g.

Proof (an outline). Using elementary arguments, the following can be established.

Fact. A continuous real-valued function defined on an open real interval is continuously differentiable if and only if it has a continuous right derivative.

Replacing $\hat{y}$ in (30) by $p\hat{x} + \hat{y}$ gives

$$d\Phi(\hat{x}, p\hat{x} + \hat{y}) = \frac{d}{dt}\bigg|_{t=p^+} \Phi(t\hat{x} + \hat{y}), \qquad \hat{x}, \hat{y} \in \hat{X}, p \geq 0. \tag{32}$$

By the above Fact and the continuity properties of Φ and $d\Phi$, for $0 < p < 1$ the right derivative in (32) is actually a two-sided derivative and it depends continuously on $\hat{y}$. Therefore, by (31), the right-hand side of (28) is equal to

[11] Note that, in the directional derivative $d\Phi$, the direction is specified by the first argument.

$$\left.\frac{d}{dt}\right|_{t=p} \Phi(tx + (1-p)y) - \left.\frac{d}{dt}\right|_{t=1-p} \Phi(px + ty).$$

By the chain rule, this expression gives the derivative on the left-hand side of (28). ■

Proposition 4 is quite useful. Many population games, like that in the following example, are themselves readily extendable to the cone of the strategy space. In addition, (31) may be more suggestive of the actual form of the potential than (28).

Example 5. *Nonatomic congestion games.* In a game g as in Example 2, the payoff function (4) is meaningful not only for (mixed) strategies, but for all $x, y \in \mathbb{R}_+^n$. The following well-known (see Milchtaich, 2004, Section 5) function $\Phi: \mathbb{R}_+^n \to \mathbb{R}$ clearly satisfies (31):

$$\Phi(x) = -\sum_{j=1}^{n} \int_0^{x_j} c_j(t)\, dt.$$

Since the cost functions are continuous and strictly increasing, Φ is continuous and strictly concave. Therefore, by Proposition 4, the restriction of Φ to the strategy space X is a strictly concave potential for g. By Theorem 4, its unique maximum point is the unique equilibrium strategy in the game and the unique stable strategy.

9 Symmetric multilinear games

Symmetric multilinear games are the N-player generalization of the two-player games considered in Section 6. The strategy space X is the unit simplex in a Euclidean space and $g: X^N \to \mathbb{R}$ is linear in each of the N arguments.

As Proposition 1 shows, stability in a symmetric multilinear game g is equivalent to stability in the population game $\bar{g}$ defined by (3). Requiring the same for *evolutionary stability* (which for a population game is given by Definition 1) yields the following natural definition.A strategy y in a symmetric multilinear game g is an ESS if, for every $x \neq y$, for sufficiently small $\epsilon > 0$ the strategy $x_\epsilon = \epsilon x + (1 - \epsilon) y$ satisfies

$$g(y, x_\epsilon, \dots, x_\epsilon) > g(x, x_\epsilon, \dots, x_\epsilon). \tag{33}$$

Strategy y is an ESS with uniform invasion barrier if it satisfies the stronger condition that, for sufficiently small $\epsilon > 0$ (which cannot vary with x), inequality (33) holds for all $x \neq y$. Note that for the existence of a uniform invasion barrier it suffices that the last condition holds for *some* $0 < \epsilon < 1$, since this automatically implies the same for all smaller ϵ.

An equivalent definition of ESS is given by a generalization of condition (iv) in Proposition 3 (Broom et al., 1997; see also the proof of Lemma 3 below).

Lemma 1. A strategy y in a symmetric multilinear game g is an ESS if and only if, for every $x \neq y$, at least one of the N terms in the sum on the left-hand side of (8) is not zero, and the first such term is negative. In particular, an ESS is necessarily an equilibrium strategy.

Unlike in the special case $N = 2$ (Proposition 3), in a general multilinear game not every ESS has a uniform invasion barrier. It is easy to see that a sufficient condition for this is that the ESS is locally superior, and this condition is in fact also necessary (Bomze and Weibull, 1995, Theorem 3; Lemma 2 below). This raises the question of how stability (in the sense of Definition 4) compares with the two nonequivalent notions of ESS. As the following theorem shows, it occupies an intermediate position: weaker than one and stronger than the other. The two ESS conditions are also comparable with the stronger stability conditions derived from $\bar{p}$-stability (see Section 4.2). In fact, two of the latter turn out to be equivalent to ESS with uniform invasion barrier.

Theorem 5. In a symmetric multilinear game, with $N \geq 2$ players, the following implications and equivalences among the possible properties of a strategy hold:

> ESS ⇐ stable ⇐ ESS with uniform invasion barrier ⇔ locally superior ⇔ dependently-stable ⇔ independently-stable ⇐ symmetrically-stable.

Each of the three implications is actually an equivalence in the special case of two-player games but not in general.

The proof of Theorem 5 uses the following two lemmas, which hold for every game g as in the theorem. The first lemma uses the following terminology. An equilibrium strategy y in g is *conditionally locally superior* if it has a neighborhood where every strategy $x \neq y$ for which (1) holds as equality satisfies inequality (7).

Lemma 2. For any $0 < p < 1$, the following properties of an equilibrium strategy y are equivalent, and imply that y is stable:

(i) Local superiority
(ii) Conditional local superiority
(iii) $\bar{p}$-stability with $\bar{p} = (p_1, p_2, \dots, p_N)$ given by (16)
(iv) $\bar{p}$-stability with $\bar{p} = (p_1, p_2, \dots, p_N)$ given by (17)
(v) ESS with uniform invasion barrier.

Proof. The implication (i) ⇒ (iii) is trivial: inequality (1) (from the equilibrium condition) and inequality (7) together give

$$(1-p)\big(g(x,y,\dots,y) - g(y,y,\dots,y)\big) + p\big(g(x,x,\dots,x) - g(y,x,\dots,x)\big) < 0.$$

Clearly, if in the last inequality the first term on the left-hand side is zero, the second must be negative. This proves that (iii) ⇒ (ii).

To prove that (ii) ⇒ (i), assume that this implication does not hold: strategy y is conditionally locally superior but not locally superior. The assumption means that there is a sequence $(x_k)_{k\geq 1}$ of strategies that converges to y such that for all k

$$g(x_k, y, \dots, y) - g(y, y, \dots, y) < 0 \tag{34}$$

and

$$g(x_k, x_k, \dots, x_k) - g(y, x_k, \dots, x_k) \geq 0. \tag{35}$$

By (34), when all the other players use y, strategy x_k is not a best response. Therefore, it can be presented as

$$x_k = \alpha_k z_k + (1 - \alpha_k) w_k, \tag{36}$$

where $0 < \alpha_k \leq 1$, z_k is a strategy whose support includes only pure strategies that are not best responses when everyone else uses the equilibrium strategy y, and w_k is a strategy that is a best response:

$$g(w_k, y, \dots, y) - g(y, y, \dots, y) = 0. \tag{37}$$

Since there are only finitely many pure strategies, there is some $\delta > 0$ such that for all k

$$g(z_k, y, \dots, y) - g(y, y, \dots, y) < -2\delta. \tag{38}$$

By (35), (36), (37) and (38), for all k

$$\big(g(x_k, x_k, \dots, x_k) - g(x_k, y, \dots, y)\big) - \big(g(y, x_k, \dots, x_k) - g(y, y, \dots, y)\big) > 2\delta\alpha_k.$$

As $k \to \infty$, the two expressions in parentheses tend to zero, since $x_k \to y$. Therefore, $\alpha_k \to 0$, which by (36) implies that $w_k \to y$. Since y is conditionally locally superior and (37) holds for all k, for almost all k (that is, all $k > K$, for some integer K) $g(w_k, w_k, \dots, w_k) - g(y, w_k, \dots, w_k) \leq 0$. Therefore, for almost all k

$$\begin{aligned}
\sum_{j=2}^{N} \frac{B_{j-1,N-1}(\alpha_k)}{\alpha_k} &\big(g(w_k, \underbrace{z_k, \dots, z_k}_{j-1 \text{ times}}, \underbrace{w_k, \dots, w_k}_{N-j \text{ times}}) - g(y, \underbrace{z_k, \dots, z_k}_{j-1 \text{ times}}, \underbrace{w_k, \dots, w_k}_{N-j \text{ times}})\big) \\
&= \frac{1}{\alpha_k}\Big(\big(g(w_k, x_k, \dots, x_k) - g(y, x_k, \dots, x_k)\big) \\
&\qquad - (1 - \alpha_k)^{N-1}\big(g(w_k, w_k, \dots, w_k) - g(y, w_k, \dots, w_k)\big)\Big) \\
&\geq \frac{1}{\alpha_k}\big(g(w_k, x_k, \dots, x_k) - g(y, x_k, \dots, x_k)\big).
\end{aligned}$$

The sum on the left-hand side tends to zero as $k \to \infty$, since $w_k \to y$. Therefore, for almost all k the expression on the right-hand side is less than δ, so that

$$g(w_k, x_k, \dots, x_k) - g(y, x_k, \dots, x_k) < \alpha_k \delta. \tag{39}$$

On the other hand, by (38) and since $x_k \to y$, for almost all k

$$\begin{aligned}
\alpha_k \Big(\big(g(y, y, \dots, y) - g(z_k, y, \dots, y)\big) &+ \big(g(z_k, y, \dots, y) \\
&- g(z_k, x_k, \dots, x_k)\big) + \big(g(w_k, x_k, \dots, x_k) - g(w_k, y, \dots, y)\big)\Big) > \alpha_k \delta.
\end{aligned}$$

By (36) and (37), the left-hand side is equal to $g(w_k, x_k, \dots, x_k) - g(x_k, x_k, \dots, x_k)$, which by (35) is less than or equal to

$$g(w_k, x_k, \dots, x_k) - g(y, x_k, \dots, x_k).$$

This contradicts (39). The contradiction proves that (ii) ⇒ (i).

To prove that (i) ⇒ (iv), assume that y is locally superior, and thus has a *convex* neighborhood U where (7) holds for every strategy $x \neq y$. By the convexity of U and the linearity of g in the first argument, for every $x \in U \setminus \{y\}$

$$g(x, x_p, \dots, x_p) - g(y, x_p, \dots, x_p) < 0, \tag{40}$$

where $x_p = px + (1 - p)y$. By the second equality in (12), (40) is equivalent to (15), with $(p_1, p_2, \dots, p_N)$ given by (17). Thus, y has property (iv).

Clearly, the above arguments also apply with p replaced by any other number in $(0,1)$. Integration over this interval therefore gives that, for every $x \in U \setminus \{y\}$, (15) holds also with $p_1, p_2, \dots, p_N$ given (not by (17) but) by the left-hand side of the corresponding equality in (13). The equalities in (13) therefore prove that the locally superior strategy y is stable.

The proof of the reverse implication (iv) ⇒ (i) is rather similar. As shown above, y has property (iv) if and only if it has a neighborhood U such that (40) holds for all strategies $x \neq y$ in U, or equivalently (7) holds for all $x \neq y$ in the set

$$U_p = \{ pz + (1-p)y \mid z \in U \}.$$

In this case, y is locally superior, since U_p is also a neighborhood of y. Indeed, for *any* neighborhood U of any strategy y, $\{U_\epsilon\}_{0<\epsilon<1}$ is a base for the neighborhood system of y (see Bomze and Pötscher, 1989, Lemma 42; Bomze, 1991, Lemma 6).

The special case $U = X$ of the last topological fact gives the equivalence (i) ⇔ (v). A strategy y has a neighborhood where (7) holds for all $x \neq y$ if and only if it has such a neighborhood of the form X_ϵ, for some $0 < \epsilon < 1$. ■

Lemma 3. For a probability vector $\bar{p} = (p_1, p_2, \dots, p_N)$ with $p_N > 0$, every $\bar{p}$-stable strategy y is an ESS.

Proof. For a vector $\bar{p}$ as above, let y be a $\bar{p}$-stable strategy and x any other strategy. For sufficiently small $0 < \epsilon < 1$,

$$\begin{aligned}
0 &> \sum_{k=1}^{N} p_k \Big(g(x_\epsilon, \underbrace{x_\epsilon, \dots, x_\epsilon}_{k-1 \text{ times}}, \underbrace{y, \dots, y}_{N-k \text{ times}}) - g(y, \underbrace{x_\epsilon, \dots, x_\epsilon}_{k-1 \text{ times}}, \underbrace{y, \dots, y}_{N-k \text{ times}})\Big) \\
&= \sum_{k=1}^{N} p_k \epsilon \sum_{j=1}^{k} B_{j-1,k-1}(\epsilon)\Big(g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(y, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}})\Big) \\
&= \sum_{j=1}^{N} \left(\sum_{k=j}^{N} \binom{k-1}{j-1} (1-\epsilon)^{k-j} p_k \right) \Big(g(x, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}}) - g(y, \underbrace{x, \dots, x}_{j-1 \text{ times}}, \underbrace{y, \dots, y}_{N-j \text{ times}})\Big) \epsilon^j.
\end{aligned}$$

This implies that, for $0 < \epsilon < 1$, at least one of the N terms on the right-hand side is not zero, and the first nonzero term (that is, the one ending with the smallest power of ϵ) is moreover negative. Since $p_N > 0$, the first expression in parentheses (i.e., the inner sum) is positive, and hence does not affect the sign of the coefficient. This proves that the condition in Lemma 1 holds. ■

Proof of Theorem 5. By Lemma 3, a strategy that has any of the seven properties in the theorem is an ESS, and hence (by Lemma 1) an equilibrium strategy. An immediate corollary of Lemma 2 is that, for an equilibrium strategy, the properties of local superiority, dependent- and independent stability, and ESS with uniform invasion barrier are equivalent, and imply stability. The special case $p = 1/2$ of the same lemma shows that symmetric-stability implies local superiority.

With only two players ($N = 2$), there is no difference between stability and symmetric-stability, and thus the equivalence of all the properties in the theorem follows from the first part of the proof and Proposition 3. The counterexamples in Example 6 below (where $N = 4$) complete the proof. ∎

Example 6. A symmetric multilinear four-player game g is defined as follows. There are three pure strategies, so that the strategy space X consists of all probability vectors $x = (x_1, x_2, x_3)$ (with $x_1 + x_2 + x_3 = 1$). The payoff of a player using strategy x against opponents using strategies $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3)$ is given by

$$g(x, y, z, w) = \sum_{i,j,k,l=1}^{3} g_{ijkl} x_i y_j z_k w_l.$$

It does not matter which of the other players uses which strategy, since the coefficients $(g_{ijkl})_{i,j,k,l=1}^{3}$ that define the game satisfy the symmetry condition $g_{ijkl} = g_{ij'k'l'}$, for all i and all triplets (j, k, l) and (j', k', l') that are permutations of one another. There are three versions of the game, with different coefficients, as detailed in the following table:

Coefficient	Version 1	Version 2	Version 3
g_{2211}	-2	-18	-4
g_{2221}	0	-16	-4
g_{3221}	4	4	0
g_{2331}	4	20	4
g_{2222}	3	-9	-3
g_{2332}	4	12	2
g_{3333}	-3	-15	-4
g_{2322}	4	4	0

Coefficients that are not listed in the table and cannot be deduced from it by using the symmetry condition are zero. In all three versions of the game, the strategy $y = (1,0,0)$ is an equilibrium strategy, since if all the other players use y, the payoff is zero regardless of the player's own strategy. However, the stability properties of y are different for the three versions.

Claim. The equilibrium strategy $y = (1,0,0)$ is an ESS in all three versions of the game, but it is stable only in versions 2 and 3, independently-stable (equivalently, dependently-stable, locally superior, ESS with uniform invasion barrier) only in version 3, and symmetrically-stable in none of them.

The Claim has some significance beyond the present context. The fact that, in version 2 of the game, the ESS $(1,0,0)$ does not have a uniform invasion barrier and is not locally superior refutes two published results. A theorem of Crawford (1990), which is reproduced by Hammerstein and Selten (1994, Result 7), implies that every ESS in a symmetric multilinear game has a uniform invasion barrier. However, there is a known error in the proof of that theorem (Bomze and Pötscher, 1993). Theorem 2 of Bukowski and Miekisz (2004) asserts that local superiority and the ESS condition are equivalent even for $N > 2$. However, these authors employ a definition of ESS that is different from that used here (and elsewhere) in that it *requires* the existence of a uniform invasion barrier.

In view of Theorem 5, to prove the Claim it suffices to show that y is: (i) an ESS but not stable in version 1, (ii) stable but not independently-stable in version 2, and (iii) independently-stable but not symmetrically-stable in version 3.

In version 1 of g, (15) reads

$$-2p_2x_2^2 - 4p_3(x_1x_2^2 - x_2^2x_3 - x_2x_3^2) \\ - 3p_4(2x_1^2x_2^2 - 4x_1x_2^2x_3 - 4x_1x_2x_3^2 - x_2^4 - 4x_2^2x_3^2 + x_3^4 - 4x_2^3x_3) < 0.$$

Stability corresponds to $\bar{p} = (p_1, p_2, p_3, p_4) = (1/4,1/4,1/4,1/4)$, for which the above inequality can be simplified to

$$\frac{7}{16}x_2^2 < (x_2 - \frac{3}{8}(1 - x_1)^2)^2.$$

There are strategies $x = (x_1, x_2, x_3)$ arbitrarily close to but different from $(1,0,0)$ for which this inequality does not hold. For example, this is so whenever $x_2 = (3/8)(1 - x_1)^2 > 0$. This proves that the equilibrium strategy is not stable. To prove that it is nevertheless an ESS, consider (33), which in the present case can be simplified to

$$2x_2^2 < (2x_2 - \epsilon(1 - x_1)^2)^2.$$

For every (fixed) strategy $x = (x_1, x_2, x_3) \neq (1,0,0)$, this inequality holds for sufficiently small $\epsilon > 0$. Therefore, $(1,0,0)$ is an ESS.

In version 2 of the game, for $\bar{p} = (1/4,1/4,1/4,1/4)$ inequality (15) can be simplified to

$$-\frac{1}{80}x_2^2 < (x_2 - \frac{3}{8}(1 - x_1)^2)^2.$$

This inequality holds for all strategies x other than $(1,0,0)$, and therefore the latter is stable. However, it is not independently-stable, since for $\bar{p} = (1/8,3/8,3/8,1/8)$ inequality (15) can be simplified to

$$\frac{1}{10}x_2^2 < (x_2 - \frac{1}{4}(1 - x_1)^2)^2.$$

This inequality does not hold for strategies x with $x_2 = (1/4)(1 - x_1)^2 > 0$, which exist in every neighborhood of $(1,0,0)$.

Finally, in version 3 of the game, for $\bar{p} = (1/8\ ,3/8,3/8,1/8)$ inequality (15) can be simplified to

$$-x_3^4 < 3(4x_2 - (x_2 + x_3)^2)^2.$$

This inequality holds for all strategies x other than $(1,0,0)$. Therefore, by Lemma 2 (which implies that, if (iv) holds for $p = 1/2$, it holds for all $0 < p < 1$), $(1,0,0)$ is independently-stable. However, it is not symmetrically-stable. There are probability vectors $\bar{p}$ satisfying (18) for which (15) does not hold for some strategies x arbitrarily close to $(1,0,0)$. For examples, for $\bar{p} = (1/20,9/20,9/20,1/20)$, inequality (15) can be simplified to

$$24x_2^2 - \frac{1}{3}x_3^4 < (8x_2 - (1 - x_1)^2)^2.$$

For strategies x with $x_2 = (1/8)(1 - x_1)^2$, this inequality is equivalent to $(1 - x_1)^4 - 32(1 - x_1)^3 + 384(1 - x_1)^2 - 2048(1 - x_1) > 512$. Hence, it does not hold if x_1 is sufficiently close to 1. This completes the proof of the Claim.

Appendix A. Other notions of stability

Static and dynamic stability are not the only kinds of stability in strategic games considered in the game-theoretic literature. For completeness, some of the other categories are briefly reviewed below.

One kind of stability refers to the effects of perturbations of the players' *strategy spaces* (e.g., allowing only completely mixed strategies) or a combination of perturbations of the strategy spaces and of the strategies themselves. The requirement that a strategy profile in a strategic game is stable against these kinds of perturbations gives the notions of (trembling-hand) perfect equilibrium (Selten, 1975), proper equilibrium (Myerson, 1978), strict perfection (Okada, 1981) and (strategic) stability and full stability (Kohlberg and Mertens, 1986). Stability may also refer to the effects on a given equilibrium of perturbations of the *payoff functions*. Essentiality (Wu and Jiang, 1962) and strong stability (Kojima et al., 1985) are examples of this kind of stability, which has interesting links with some of the other kinds. For example, in multilinear games, every essential equilibrium is strictly perfect (van Damme, 1991, Theorem 2.4.3), and in symmetric $n \times n$ games, every regular ESS is essential (Selten, 1983). Another example of a link between different kinds of stability is the finding that, in several classes of games, the (local) degree of an equilibrium (or of a connected component of equilibria) is equal to its index (Govindan and Wilson, 1997; Demichelis and Germano, 2000). The index of an equilibrium is connected with its asymptotic stability or instability with respect to a large class of natural dynamics, which determine how strategies in the game change over time. The degree, by contrast, expresses a topological property of the equilibrium when viewed as a point in a manifold that includes the various equilibria of *different* games (Ritzberger, 2002). The index (= degree) of an equilibrium is connected with stability also in that, in a nondegenerate bimatrix game, it determines whether it can be made the unique equilibrium by extending the game, adding one or more pure strategies to one of the players (von Schemde and von Stengel, 2008).

Whether any of these alternative notions can be linked with statics stability is yet to be determined.

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